A Second Course in String Theory

Conformal Field Theory, Superstrings, Duality and Twistors

Edward Hughes, Queen Mary University of London

These notes are intended for the beginning PhD student in string theory or related fields of theoretical physics. They provide a lightning overview of several important areas in contemporary research, with no pretense at full explanation or rigour. We hope that they might provide sufficient information to mitigate the daunting complexity of academic seminars. As such, they provide a useful reference, and humble starting block for a full and thorough exploration of the literature.

We assume that the reader has completed a Masters level qualification in theoretical physics of comparable standard to the Cambridge Part III Mathematics course. In particular, the intended audience should have basic proficiency with bosonic string theory and elementary differential geometry. As a refresher on these topics we can recommend no better resources than the following.

- David Tong, *Lecture Notes on String Theory*

Taken from his renowned Part III course, these notes introduce the fundamental principles and tools of modern string theory in David Tong's characteristically clear and engaging style. Available online at `http://www.damtp.cam.ac.uk/user/tong/string.html`.

- Nakahara, *Geometry, Topology and Physics*

Quite simply the reference bible for elementary differential geometry and its applications in contemporary theoretical physics. For this course, it should suffice to review Chapter 5 on Manifolds. If your PhD office is missing this textbook, ask your supervisor to purchase it post-haste!

Contents

1 Conformal Field Theory

We recap the rudiments of conformal field theory, often forgotten by students yet omitted from seminars.

1.1 Conformal Symmetry

In $d > 2$ the **conformal algebra** is $\frac{1}{2}(d+1)(d+2)$-dimensional with generators $p^\mu, J^{\mu\nu}, D$ and K^μ generating translations

$$e^{ia\cdot p} x e^{-ia\cdot p} = x - a \tag{1.1}$$

rotations or boosts

$$e^{ir\cdot J} x e^{-ir\cdot J} = Lx \qquad \text{with } L \in SO(1,3)^+ \tag{1.2}$$

dilatations

$$e^{icD} x e^{-icD} = e^{-c} x \tag{1.3}$$

and special conformal transformations

$$e^{i\alpha K} \frac{x}{x^2} e^{-i\alpha K} = \frac{x}{x^2} - \alpha \tag{1.4}$$

Aside from the usual Lorentz algebra commutators, we have in addition

$$[J_{\mu\nu}, K^\lambda] = i\delta^\lambda_{[\mu} K_{\nu]}\,, \quad [J_{\mu\nu}, D] = 0\,, \quad [P^\mu, K^\nu] = -2i\eta^{\mu\nu} D + 2iJ^{\mu\nu}\,, \quad [P^\mu, D] = iP^\mu\,, \quad [D, K^\lambda] = iK^\lambda \tag{1.5}$$

In $d = 2$ the **conformal algebra** is infinite-dimensional with generators

$$L_n = -z^{n+1} \frac{\partial}{\partial z} \tag{1.6}$$

for $n \in \mathbb{Z}$. The commutation relations are

$$[L_m, L_n] = (m-n) L_{m+n} \tag{1.7}$$

A **conformal field theory** is a quantum field theory whose quantum effective action is invariant under conformal transformations. In $d = 2$ dilatation invariance suffices to guarantee conformal invariance, but the analogous statement in generic dimension is as yet undecided.

In $d = 2$ conformal invariance as defined above is typically too strong a condition for interesting theories, due to a generic Weyl anomaly. We thus extend the definition of conformal field theory to include the case where the quantum effective action is invariant under Virasoro transformations, generated by the L_n above and a central charge c such that

$$[c, L_n] = 0 \quad \text{and} \quad [L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{1.8}$$

A key observable in a scale invariant quantum field theory is the **scaling dimension** Δ of an operator. More precisely, scale invariance of the effective action is equivalent to vanishing of the β function, hence under dilatations a generic operator transforms as

$$e^{i\lambda D}\mathcal{O}(x)e^{-i\lambda D} = \lambda^{-\Delta}\mathcal{O}(\lambda x) \tag{1.9}$$

where Δ does not depend on the distance scale. In particular, the scaling dimensions fix the two-point function up to some operator-dependent constant c_{12}

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{c_{12}\delta_{\Delta_1-\Delta_2,0}}{|x_{12}|^{\Delta_1+\Delta_2}} \tag{1.10}$$

This observation (and its three-point cousin) forms the starting point for the fruitful conformal bootstrap programme. In this line of work, it is common to come across quantities which depend on the difference between scaling dimension and Lorentz spin. Accordingly, this quantity has been christened **twist** by the community.

1.2 State-Operator Correspondence

In $d = 2$ we may use conformal symmetry to map the cylinder with coordinates (σ, τ) to the Riemann sphere with coordinates $(z, \bar{z})$ via $z = e^{\sigma+i\tau}$. Under this map time-ordering for correlation functions becomes radial ordering; that is to say that the dilatation operator governs the evolution of the system. Canonical quantization in this setting is known as **radial quantization**, for obvious reasons.

The true power of this perspective is to make manifest the fact that states and local operators are in bijection in a $d = 2$ conformal field theory in a manner compatible with their appearance in correlation functions[1]. To prove this it is easiest to work in the language of second quantisation, where states Ψ are functionals of field configurations ϕ_b on circles around the origin. We may map from an operator insertion in the far past to a state by performing a path integral with boundary condition $\phi_i = \phi_b$, viz.

$$\Psi[\phi_b] = \int \mathcal{D}\phi_i e^{-S[\phi_i]}\mathcal{O}(0) \tag{1.11}$$

Conversely we define an operator $O(0)$ corresponding to Ψ by taking the limit of a path integral over progressively smaller circles with radius r and field configuration ϕ_c, viz.

$$\mathcal{O}(0) = \lim_{r\to 0}\int \mathcal{D}\phi_c r^{-D}\Psi[\phi_c] \tag{1.12}$$

This constitutes the inverse of the previous equation since r^D is precisely responsible for propagating states from the circle $|z| = r$ to the boundary.

For practical purposes this proof is rather cumbersome. In string theory, we typically require to construct a so-called **vertex operator** corresponding to a state for use in heuristic perturbative formulae, for which full derivations are not known owing to the complexity of string field theory. To obtain such operators, we observe that the correspondence above preserves the transformation properties of states

[1]The state-operator correspondence is also valid in $d > 2$ but the construction is more subtle.

of operators under the action of symmetries. In simple cases, it is then usually possible to bootstrap the correct vertex operator.

First observe that vertex operators must be gauge invariant, i.e. unchanged under worldsheet reparameterisation. The natural such objects are $(1,1)$ forms integrated over the worldsheet Σ, viz.

$$V = \int_\Sigma \mathcal{O}\ dz \wedge d\bar{z} \tag{1.13}$$

Suppose we want V to describe a tachyon of momentum k for a bosonic closed string. Then under a spacetime translation $X \to X + a$ the operator must transform with weight $e^{ik\cdot a}$. Moreover, it must be a Lorentz scalar. We are thus naturally led to conjecture

$$V_{\text{tachyon}} = \int_\Sigma :e^{ik\cdot X}: dz \wedge d\bar{z} \tag{1.14}$$

More generally one must include an additional polynomial in X^μ and its derivatives which is a worldsheet scalar and has the correct Lorentz index structure matching the spectrum most easily derived via lightcone-gauge quantization.

1.3 Operator Product Expansion

It is conjectured that in any quantum field theory products of local operators at different points may be expanded as a linear combination of operators at one of the points

$$A(x)B(y) = \sum_i c_i |x-y|^i C_i(y) \tag{1.15}$$

where c_i are non-universal coefficients. Some care is needed to interpret this statement. Firstly, we must view the right-hand-side as an asymptotic series in $|x-y|$. Secondly, the expressions should be equated in the sense of operator insertions within time-ordered correlation functions. These mathematical subtleties place no great restrictions on the physics, however; after all, correlation functions encapsulate all observable quantities, and we are well-used to dealing with asymptotic series from perturbation theory.

While the operator product expansion was first used to study the hadronic e^+e^- cross-section, it is particularly powerful in $d=2$ conformal field theory. In this case, the operator product expansion is fully proven as a convergent series by virtue of the state-operator correspondence. Explicitly we find

$$A(z_1,\bar{z_1})B(z_2,\bar{z}_2) = \sum_k z_{12}^{h_k - h_i - h_j} \bar{z}_{12}^{\bar{h}_k - \bar{h_i} - \bar{h_j}} c_i C_i(z_2,\bar{z}_2) \tag{1.16}$$

where we assume wlog that A and B are eigenstates under the scaling $z \to \lambda z$, hence transforming like

$$A'(\lambda z, \bar{\lambda}\bar{z}) = \lambda^{-h}\bar{\lambda}^{-\bar{h}} A(z,\bar{z}) \tag{1.17}$$

where $(h,\bar{h})$ are independent quantities called **conformal weights**. It is not hard to derive that $\Delta = h + \bar{h}$ and Lorentz spin is $h - \bar{h}$, so that $\bar{h}$ encodes the twist of the operator.

For practical purposes we may determine parts of the OPE by using the Ward identities associated with conformal symmetry. Indeed from the path integral approach to QFT we are familiar with identities such as

$$\partial_\mu \langle j^\mu(x)\phi_{a_1}(x_1)\dots\phi_{a_n}(x_n)\rangle = -i\sum_{j=1}^{n}\langle\phi_{a_1}(x_1)\dots\delta^{(n)}(x-x_j)\delta\phi_{a_j}(x_j)\dots\phi_{a_n}(x_n)\rangle \tag{1.18}$$

In $d=2$ we can squeeze extra information out of such identities by integrating both sides over $\mathcal{C}_\infty$ and using Stokes' theorem to convert to a contour integral. Now splitting the current as a sum of holomorphic and antiholomorphic pieces yields

$$\frac{i}{2\pi}\oint j(z)\mathcal{O}(w,\bar{w}) = \delta\mathcal{O}(w,\bar{w}) \tag{1.19}$$

and similarly for $\bar{j}(\bar{z})$. In other words, the transformation of $\mathcal{O}(w,\bar{w})$ under a symmetry determines the coefficient of the simple pole in $(z-w)$ in its OPE with the conserved currents $j(z)$ and $\bar{j}(\bar{z})$.

It is convenient to define a class of operators for which the conformal Ward identies determine the singular behaviour of the OPEs completely. These are the **primary operators**, whose OPEs with the energy-momentum tensor $T(z,\bar{z}) = T(z)+\bar{T}(\bar{z})$ have at worst a double pole. Examining the behaviour of a generic operator $\mathcal{O}(w,\bar{w})$ under translations, rotations and scaling leads to

$$T(z)\mathcal{O}(w,\bar{w}) = h\frac{O(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\mathcal{O}(w,\bar{w})}{(z-w)} + \dots \tag{1.20}$$

and similarly for $\bar{T}(\bar{z})$. Primary operators are the basic building blocks of a $d=2$ CFT, in the sense that OPEs for any operators may be derived from OPEs for primary operators.

In string theory the operator product expansion has twofold significance. On the one hand it provides a convenient way to describe constraints on correlation functions from Ward identities. On the other, it may be employed as a calculational tool to determine the normal ordered product of vertex operators required for string amplitudes.

2 Superstrings

Most young researchers have an intimate knowledge of the bosonic string. Most current research requires an understanding of the superstring. Here we bridge the gap.

2.1 RNS Action

In bosonic string theory we generalise the action for a massless particle

$$S = \frac{1}{2}\int d\tau \ \left(e^{-1}\dot{x}^\mu \dot{x}_\mu\right) \tag{2.1}$$

where $e(\tau)$ is an auxiliary field, mathematically required for reparameterisation invariance

$$\delta_\xi x^\mu = \xi \dot{x}^\mu \,, \quad \delta_\xi e = \partial_\tau(\xi e) \tag{2.2}$$

where $\xi(\tau)$ is a local infinitesimal parameter. The two-dimensional version of this is the **Polyakov action**

$$S = \frac{T}{2}\int d^2\sigma \ \left(\sqrt{h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right) \tag{2.3}$$

where now the metric $h_{\alpha\beta}$ is an auxiliary field (with determinant h), which can be completely fixed to $\eta_{\alpha\beta}$ by reparameterising the worldsheet

$$\delta_\xi X^\mu = \xi^\alpha \partial_\alpha X^\mu \,, \quad \delta_\xi h^{\alpha\beta} = \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^\alpha h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma} \tag{2.4}$$

and applying a Weyl scaling

$$\delta_\Lambda X^\mu = 0 \,, \quad \delta_\Lambda h^{\alpha\beta} = \Lambda h^{\alpha\beta} \tag{2.5}$$

To supersymmetrise this story, it is natural to begin with the action for a massless particle with spin degrees of freedom, namely

$$S = \frac{1}{2}\int d\tau \ \left(e^{-1}\dot{x}^2 + \psi^\mu \dot{\psi}_\mu - e^{-1}\chi\psi^\mu \dot{x}_\mu\right) \tag{2.6}$$

where ψ^μ_A is a D-plet of worldsheet Majorana spinors, and χ_A is a Majorana auxiliary field, necessary for local supersymmetry. Indeed, it is an easy exercise to show invariance under the transformations

$$\delta_\epsilon x^\mu = \epsilon\psi^\mu \,, \quad \delta_\epsilon e = \epsilon\chi \,, \quad \delta_\epsilon \psi^\mu = \epsilon\left(\frac{1}{2}\chi\psi^\mu - \dot{x}^\mu\right)e^{-1} \,, \quad \delta_\epsilon \chi = 2\dot{\epsilon} \tag{2.7}$$

where $\epsilon(\tau)$ is a local infinitesimal parameter. This model was originally constructed to reproduce the Dirac equation upon quantization, in the same way that (2.1) yields the Klein-Gordon equation. Hence it is clear that (2.6) does not possess spacetime supersymmetry.

The two-dimensional version is the **Brink-di Vecchia-Howe** action[2]

$$S = \frac{T}{2}\int d^2\sigma \ \det(e)\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu + 2\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \psi^\mu \partial_\beta X_\mu + \frac{1}{2}(\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \chi_\beta)(\bar{\psi}^\mu \psi_\mu)\right) \tag{2.8}$$

[2]As usual for a Majorana spinor $\bar{\psi}$ denotes $\psi^\mathsf{T}\gamma^0$.

In addition to the auxiliary metric $h^{\alpha\beta}$, we now have supergravity multiplet $(e^a_\alpha, \chi_\alpha)$ of Lagrange multipliers comprising a zweibein and two Majorana fermions. In particular the zweibein is responsible for defining gamma matrices compatible with general covariance, via $\gamma^\alpha = \gamma^a e^\alpha_a$.

This is invariant under worldsheet reparameterisations, local supersymmetry transformations

$$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu \,, \quad \delta_\epsilon \psi^\mu = i\left(\partial_\alpha X^\mu + \frac{1}{4}\bar{\chi}_\alpha \psi^\mu\right)\gamma^\alpha \epsilon \,, \quad \delta_\epsilon e^a_\alpha = -2i\bar{\epsilon}\gamma^a \chi_\alpha \,, \quad \delta_\epsilon \chi_\alpha = D_\alpha \epsilon \tag{2.9}$$

and super-Weyl rescalings

$$X^\mu \to X^\mu \,, \quad \psi^\mu \to \Lambda^{-\frac{1}{2}}\psi^\mu \,, \quad e^a_\alpha \to \Lambda e^a_\alpha \,, \quad \chi_\alpha \to \Lambda^{\frac{1}{2}}\chi_\alpha + i\gamma_\alpha \lambda \tag{2.10}$$

where D is the covariant derivative of a spinor in two dimensions, λ is an arbitrary Majorana spinor and ϵ^A is a Majorana spinor of Grassmann variables. The presence of so many auxiliary fields makes (2.8) rather impractical. Fortunately one can gauge fix to obtain the much simpler **Ramond-Neveu-Schwarz** action

$$S = \frac{T}{2}\int d^2\sigma \ \left(\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - i\bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu\right) \tag{2.11}$$

looking like the free theory of a Nambu-Goto string and several Majorana fermions. The equations of motion derived from (2.11) must still be supplemented with the constraints imposed by the Lagrange multipliers (h, e, χ) to obtain the correct physical degrees of freedom.

It is highly non-trivial that (2.11) supplemented with gauge consistency conditions yields a spacetime supersymmetric theory upon quantisation. To see this rigorously, it is easiest to start with an equivalent action due to Green and Schwarz with manifest spacetime symmetry. More explicitly, one may use residual gauge invariance to fix **lightcone gauge**[3]

$$X^+(\sigma,\tau) = x^+ + p^+\tau \,, \quad \psi^+ = 0 \tag{2.12}$$

and then perform a field redefinition, recasting the RNS action into GS form. The main drawback of the GS approach is the difficulty of covariant quantisation due to the appearance of second-class constraints.

2.2 Boundary Conditions

To canonically quantise the string, we require the classical solutions for X^μ and ψ^μ. We must first derive the equations of motion and boundary conditions by requiring a stationary action under variation of the dynamical fields. For the bosonic part of (2.11) we find[4]

$$\begin{aligned} 0 = 2\pi\alpha'\delta S &= \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \ \partial_\alpha X \cdot \partial^\alpha \delta X \\ &= -\int d^2\sigma \ (\partial^\alpha\partial_\alpha X)\cdot \delta X - \left[\int_0^\pi d\sigma \ \dot{X}\cdot\delta X\right]_{\tau=\tau_i}^{\tau=\tau_f} + \left[\int_{\tau_i}^{\tau_f} d\tau \ X'\cdot \delta X\right]_{\sigma=0}^{\sigma=\pi} \end{aligned} \tag{2.13}$$

[3]The spacetime lightcone coordinates are defined to be $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$.

[4]We employ the standard notation $X' = \partial_\sigma X$ and $\dot{X} = \partial_\tau X$.

The bulk term yields the free wave equation

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = 0 \tag{2.14}$$

with general solution[5]

$$X^\mu(\sigma,\tau) = X_R^\mu(\sigma_-) + X_L^\mu(\sigma_+) \tag{2.15}$$

where the right-moving and left-moving excitations are independent arbitrary functions. The first boundary term vanishes identically by requiring $\delta X = 0$ at τ_i and τ_f as usual for an action principle. The second boundary term yields additional conditions, absent for point particles. These come in two topological flavours, **closed strings**

$$X^\mu|_{\sigma=0} = X^\mu|_{\sigma=\pi} \tag{2.16}$$

and **open strings**

$$X'^\mu|_{\sigma=0,\pi} = 0 \text{ (Neumann)} \quad \text{or} \quad X^\mu|_{\sigma=0,\pi} = c^\mu \text{ (Dirichlet)} \tag{2.17}$$

In the early days of string theory the Neumann boundary conditions were preferred for open strings since they ensure that spacetime momentum[6] doesn't cross the $\sigma = 0, \pi$ boundary of the worldsheet. Indeed, we may verify this directly by considering the local transformation

$$X^\mu(\sigma^\alpha) \to X^\mu(\sigma^\alpha) + \epsilon^\mu(\sigma^\alpha) \tag{2.18}$$

yielding (via Noether's trick) the momentum current

$$P_\alpha^\mu = T\partial_\alpha X \tag{2.19}$$

The momentum flowing across a line segment $(0, d\tau)$ or $(\pi, d\tau)$ at an open string endpoint is then

$$dP^\mu = dP_\sigma^\mu d\tau = TX'^\mu \tag{2.20}$$

which exactly vanishes under Neumann boundary conditions.

Nowadays we also allow Dirichlet boundary conditions, intepreting the "lost" momentum as belonging to some extended object fixed at c^μ for whichever μ are Dirichlet. For this to make sense the string must couple to the object in a consistent way, which places constraints on which combinations of Dirichlet and Neumann conditions are allowed, or equivalently which extended objects are stable. We shall return to this discussion in Section 3.

[5] The worldsheet lightcone coordinates are defined to be $\sigma^\pm = \tau \pm \sigma$.

[6] We distinguish this from the worldsheet momentum, which cannot flow across the string boundary courtesy of Cardy's condition for CFTs with boundaries.

The general solutions for closed strings may be written[7]

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\sigma^- p^\mu + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-} \tag{2.21}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\sigma^+ p^\mu + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+} \tag{2.22}$$

where α_n^μ and $\tilde{\alpha}_n^\mu$ are independent oscillators. For open strings we find

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\sigma^- p^\mu + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-} \tag{2.23}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\sigma^+ p^\mu + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+} \tag{2.24}$$

where now the oscillator modes are not independent, rather

$$\alpha_n^\mu = \tilde{\alpha}_n^\mu \text{ (Neumann)}\,, \quad \alpha_n^\mu = -\tilde{\alpha}_n^\mu\,,\ x^\mu = c^\mu\,,\ p^\mu = 0 \text{ (Dirichlet)} \tag{2.25}$$

While much of this material may be familiar from a first course in string theory, we have presented it in a manner which easily generalises to the RNS action (2.11). For clarity, we work in the basis of γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.26}$$

and write ψ^μ in spinor components as

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix} \tag{2.27}$$

Varying ψ^μ and requiring a stationary action yields

$$0 = 2i\pi\alpha'\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma\ \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \delta\psi_\mu \tag{2.28}$$

$$= -\int d^2\sigma\ \partial_\alpha \bar{\psi}^\mu \gamma^\alpha \delta\psi_\mu + \left[\int_0^\pi d\sigma\ \bar{\psi}^\mu\gamma^0\delta\psi_\mu\right]_{\tau=\tau_i}^{\tau=\tau_f} + \left[\int_{\tau_i}^{\tau_f} d\tau\ \bar{\psi}^\mu\gamma^1\delta\psi_\mu\right]_{\sigma=0}^{\sigma=\pi} \tag{2.29}$$

The first term gives the (conjugate) Weyl equation in each spacetime component of ψ^μ, with general solution

$$\psi^\mu(\sigma,\tau) = \begin{pmatrix} \psi_-(\sigma^-) \\ \psi_+(\sigma^+) \end{pmatrix} \tag{2.30}$$

The second term vanishes by standard boundary conditions for action principles. The third term yields the additional boundary condition

$$\psi^\mathsf{T}\gamma^0\gamma^1 \cdot \delta\psi = 0 \quad \text{i.e.} \quad \psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_- = 0 \tag{2.31}$$

at the endpoints $\sigma = 0, \pi$. As for the bosonic string, there are two classes of solutions, closed strings

$$\psi_+|_{\sigma=0} = \pm\psi_+|_{\sigma=\pi}\,, \quad \psi_-|_{\sigma=0} = \pm\psi_-|_{\sigma=\pi} \tag{2.32}$$

[7]Henceforth we use natural units in which $\sqrt{\pi T} = 1$. The reader may reinsert such factors by dimensional analysis.

where the choice of $\pm$ is independent in ψ_+ and ψ_-, and open strings

$$\psi_+|_{\sigma=0} = \psi_-|_{\sigma=0}\ , \quad \psi_+|_{\sigma=\pi} = \pm\psi_-|_{\sigma=\pi} \tag{2.33}$$

where we wlog fix the relative sign at one endpoint of the string.

We assign names to the boundary conditions according to which one of $\pm$ is chosen: **Ramond (R)** boundary conditions have a $+$ sign and **Neveu-Schwarz (NS)** have a $-$ sign. Hence open superstrings come in R or NS varieties, wheareas closed superstrings can be R-R, R-NS, NS-R or NS-NS.

Finally, we collect the general solutions for the fermionic components of the superstring. For closed strings, we may write

$$\begin{aligned} 2\psi_-^\mu &= \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in\sigma^-} \qquad \text{or} \qquad \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2ir\sigma^-} \\ \psi_+^\mu &= \sum_{n\in\mathbb{Z}} \tilde{d}_n e^{-2in\sigma^+} \qquad \text{or} \qquad \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2ir\sigma^+} \end{aligned} \tag{2.34}$$

where $(d_n, \tilde{d}_n)$ are associated with Ramond boundary conditions, while $(b_r, \tilde{b}_r)$ are associated with Neveu-Schwarz boundary conditions. For open strings we have, employing analogous notation

$$\psi_-^\mu = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^-} \qquad \text{or} \qquad \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{2ir\sigma^-} \tag{2.35}$$

$$\psi_+^\mu = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n e^{-in\sigma^+} \qquad \text{or} \qquad \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir\sigma^+} \tag{2.36}$$

2.3 Constraints

As argued in Section 2.1, it is not valid simply to solve the equation of motion following from the RNS action (2.11). We must remember that this is the gauge fixed form of the Brink-di Vecchia-Howe action (2.8), possessing Lagrange multipliers. These impose constraints which must be taken into account.

For the bosonic part of the action, the constraints are easy to derive. By Noether's trick, recall that the variation of the action upon changing the metric exactly yields the energy-momentum tensor $T_{\alpha\beta}$. The reader may explicitly vary the metric in the Polyakov action to derive

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X \tag{2.37}$$

in the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$. The constraints are therefore just $T_{\alpha\beta} = 0$ for X on-shell, or in lightcone coordinates

$$T_{++} = \partial_+ X \cdot \partial_+ X = 0\,, \quad T_{--} = \partial_- X \cdot \partial_- X = 0 \tag{2.38}$$

with $T_{+-} = T_{-+} = 0$ already automatic from the tracelessness of (2.37).

We can pursue a similar programme for the Brink-di Vecchia-Howe action. By Noether's trick, the energy momentum tensor has non-zero components[8]

$$T_{++} = \partial_+ X \cdot \partial_+ X + \frac{i}{2}\psi_+ \cdot \partial_+ \psi_+ \,, \quad T_{--} = \partial_- X \cdot \partial_- X + \frac{i}{2}\psi_- \cdot \partial_- \psi_- \tag{2.39}$$

and as above, varying h implies the constraint $T_{++} = T_{--} = 0$. We may similarly derive the nonzero components of the conserved current associated to global supersymmetry[9]

$$J_{++} = \psi_+ \cdot \partial_+ X \,, \quad J_{--} = \psi_- \cdot \partial_- X \tag{2.40}$$

It is an elementary exercise to show that the Lagrange multipliers (e, χ) lead to the constraint equations $J_{++} = J_{--} = 0$, as we might have guessed.

It will be convenient to express the energy-momentum constraints as the vanishing of the **Virasoro modes**[10] obtained as the Fourier components of T_{--} and T_{++}. For open strings we define

$$L_m = \frac{1}{\pi}\int_0^\pi \left(e^{2im\sigma}T_{++} + e^{-2im\sigma}T_{--}\right) d\sigma = 0 \tag{2.41}$$

or in terms of oscillator modes[11],

$$L_m^{\mathrm{NS}} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n + \frac{1}{2}\sum_{r=-\infty}^{\infty}\left(r + \frac{1}{2}m\right) b_{-r}\cdot b_{m+r} = 0 \tag{2.42}$$

$$L_m^{\mathrm{R}} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n + \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(n + \frac{1}{2}m\right) d_{-n}\cdot d_{m+n} = 0 \tag{2.43}$$

Closed strings have additional sets of constraints for the $\tilde{\alpha}$ modes. Observe that we have imposed the constraints at $\tau = 0$ without loss of information since $T_{\alpha\beta}$ is a conserved current.

The $L_0 = 0$ constraint is particularly important, since it determines the mass of the string in terms of its internal modes of oscillation. Indeed defining $M^2 = -p_\mu p^\mu$ we find, courtesy of footnote 11,

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty} n d_{-n}\cdot d_n\right) \tag{2.44}$$

for open Ramond strings, with similar formulae in other sectors.

2.4 Quantisation and Spectrum

We will now quantize the RNS string in the lightcone gauge (2.12). This fixes all the gauge symmetry of the RNS action (2.11) at the expense of manifest Lorentz covariance. The advantage of this approach

[8]Strictly speaking the following expressions are valid in the gauge that produces the RNS action.

[9]Recall that the supersymmetry parameter is a Majorana spinor of Grassmann variables with components ϵ^A. Hence the current has components $J_{\alpha A}$. We may write $\alpha = \pm$ courtesy of lightcone coordinates and $A = \pm$ analogously to (2.27).

[10]It is no coincidence that the notation in (2.41) agrees with (1.6) – indeed, the Virasoro modes precisely generate worldsheet reparameterisations in the Poisson bracket sense.

[11]We adopt the useful convention $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2}p^\mu$ for closed strings, and $\alpha_0^\mu = p^\mu$ for open strings.

is that we may solve the constraints (2.38) completely, so the spectrum contains no ghosts and we may immediately read off the physical states.

The first step in **canonical quantisation** is to impose the equal-time (anti)commutation relations

$$\begin{aligned} [X^\mu(\sigma,\tau), P_\nu(\sigma',\tau)] &= i\delta(\sigma-\sigma')\delta^\mu_\nu \\ \{\psi^\mu_A(\sigma,\tau), \psi_{B\nu}(\sigma',\tau)\} &= \pi\delta(\sigma-\sigma')\delta^\mu_\nu\delta_{AB} \end{aligned} \tag{2.45}$$

which immediately imply for the Fourier modes

$$\begin{aligned} [\alpha^\mu_m, \alpha_{n\nu}] &= m\delta_{m+n}\delta^\mu_\nu \\ [\tilde\alpha^\mu_m, \tilde\alpha_{n\nu}] &= m\delta_{m+n}\delta^\mu_\nu \\ \{b^\mu_r, b_{s\mu}\} &= \delta^\mu_\nu\delta_{r+s} \\ \{d^\mu_m, d_{n\nu}\} &= \delta^\mu_\nu\delta_{m+n} \end{aligned} \tag{2.46}$$

By comparison with point-particle quantum field theory, we may immediately identify $\alpha_m, \tilde\alpha_m, b_r, d_m$ as raising operators for $m, r > 0$ and lowering operators for $m, r < 0$.

When translating fields into operators via canonical quantisation, one occasionally encounters **operator ordering ambiguities**, which must be fixed by some consistency condition. For instance, the Hamiltonian of a quantum field theory is usually taken to be normal ordered, with the (partial) justification that only differences in energy are measurable.

We encounter just such an ambiguity when promoting L_0 to an operator, since the oscillators appearing therein have non-trivial commutation relations. To resolve this we define L_0 by requiring it to be implicitly normal ordered, and then include an arbitrary constant in the related constraint

$$(L_0 - a)|\phi\rangle = 0 \tag{2.47}$$

In particular the mass formula (2.44) gains a constant shift at the quantum level. In principle a is different in the Ramond and Neveu-Schwarz sectors.

In lightcone gauge the constraints (2.39) and (2.40) reduce to[12]

$$\partial_+ X^-_L = \frac{1}{p^+}\left(\partial_+ X^i_L \partial_+ X^i_L + \frac{i}{2}\psi^i_+\partial_+\psi^i_+\right) \tag{2.48}$$

$$\psi^-_+ = \frac{2}{p^+}\psi^i_+\partial_+ X^i_L \tag{2.49}$$

and analogous equations with replacements $+ \to -$ and $L \to R$ in the lower indices. Hence we can solve for X^- and ψ^- up to a constant. In particular, one can show that α^-_0 and d^-_0 have normal ordering

[12]The index $i = 1, \ldots D-2$ ranges over the so-called transverse coordinates of the string. This moniker is customary yet misleading, given that the string doesn't necessarily live in the X^0–X^{D-1} plane.

ambiguities, resulting in the appearance of the constant a as in (2.47). All the dynamics now resides in the transverse oscillators $\alpha^i_m, \tilde{\alpha}^i_m, b^i_r, d^i_m$.

Before we can determine the mass spectrum of the RNS superstring, we must fix the operator ordering ambiguity a. This emerges from requiring quantum consistency of the theory. More precisely, fixing X^+ and ψ^+ and solving for X^- and ψ^- breaks manifest Lorentz symmetry. Carefully defining the spacetime Lorentz generators $J^{\mu\nu}$ in terms of superstring modes and requiring that $[J^{i-}, J^{j-}] = 0$ enforces

$$D = 10\,, \quad a_{\mathrm{NS}} = \frac{1}{2}\,, \quad a_{\mathrm{R}} = 0 \tag{2.50}$$

In covariant path integral quantisation, these values emerge from requiring that the conformal gauge symmetry of the worldsheet remains unbroken at the quantum level.

We are now well-placed to determine the lightest states in the superstring spectrum. Of course, we must first identify the **ground states** for each type of superstring.

For a Neveu-Schwarz open string we define $|0;k\rangle_{\mathrm{NS}}$ by

$$\alpha^i_n|0;k\rangle_{\mathrm{NS}} = b^i_r|0;k\rangle_{\mathrm{NS}} = 0 \quad \text{for } n, r > 0\,, \quad \alpha^i_0|0;k\rangle_{\mathrm{NS}} = \sqrt{2\alpha'}k^i|0;k\rangle_{\mathrm{NS}} \tag{2.51}$$

which is clearly a state of lowest mass[13], by virtue of the quantum version of (2.44) in the NS sector and lightcone gauge, namely

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + \sum_{r=1/2}^{\infty} r b^i_{-r}b^i_r - \frac{1}{2}\right) \tag{2.52}$$

Since $|0;k\rangle_{\mathrm{NS}}$ carries momentum and has no Lorentz indices, it is clearly a spacetime scalar particle, with mass-squared $-1/(2\alpha')$. Alas! This state is therefore tachyonic, just as in the bosonic string. Fortunately we shall shortly see that it must be excised from the spectrum in a well-defined manner for a consistent interacting theory. The first excited state is $b_{-1/2}|0;k\rangle_{\mathrm{NS}}$ a spacetime vector gauge boson with mass-squared 0, consistent with Lorentz symmetry. More generally, one may easily observe that all states of the Neveu-Schwarz open string will be **bosonic**.

In the Ramond sector we have ground state

$$\alpha^i_n|0;k\rangle^a_{\mathrm{R}} = d^n_r|0;k\rangle^a_{\mathrm{R}} = 0 \quad \text{for } n > 0\,, \quad \alpha^i_0|0;k\rangle^a_{\mathrm{R}} = \sqrt{2\alpha'}k^i|0;k\rangle^a_{\mathrm{R}} \tag{2.53}$$

where we have suggestively included a spinor index a. Indeed this is required because the d^i_0 modes commute with the Ramond sector mass-squared operator, which is precisely (2.44). The d^i_0 modes have anticommutation relations $\{d^i_0, d^j_0\} = \eta^{ij}$, so our states at every mass level must furnish representations of the Clifford algebra. In particular, the ground state must be an irreducible representation, since there

[13]And therefore lowest energy for $k = 0$.

are no other zero-modes that would cause any degeneracy.

Therefore the ground state is a spinor of $SO(1,9)$ with mass-squared 0. The first excited states are $\alpha^i_{-1}|0;k\rangle_{\mathrm{R}}$ and $d^i_{-1}|0;k\rangle_{\mathrm{R}}$, spin 3/2 gauginos with mass-squared $1/\alpha'$. More generally, all states of the Ramond open string will be **fermionic**.

In fact, this isn't the full story of the open superstring spectrum. Loop-level consistency of the interacting theory disallows some states. More explicitly, modular invariance of the string worldsheet, trivial at genus 0 but required for tori, enforces a **Gliozzi-Scherk-Olive projection**. For example, in the open NS sector define an operator

$$G_{\mathrm{NS}} = (-1)^{\sum_{r=1/2}^{\infty} b^i_{-r} b^i_r + 1} \tag{2.54}$$

The GSO projection is defined to eliminate all states with an eigenvalue of -1 under the action of G_{NS}. In particular the tachyon $|0;k\rangle_{\mathrm{NS}}$ is removed, and the ground state of the open NS string becomes a gauge boson $b^i_{-1/2}|0;k\rangle_{\mathrm{NS}}$. One may define a similar operator in the Ramond sector, namely

$$G_{\mathrm{R}} = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d^i_{-n} d^i_n} \tag{2.55}$$

where $\Gamma_{11} = \Gamma_0 \dots \Gamma_9$ is the ten-dimensional version of the familiar γ_5 matrix in four dimensions. Since the Ramond sector does not contain a tachyon, it is not obvious whether we should project onto states G_R eigenvalue $+1$ or -1. It turns out that modular invariance is valid for either alternative, so the selection is simply a matter of convention. We index our choice by considering the action of G_{R} on the ground state gaugino, deducing that it must have definite (positive or negative) chirality.

Now we are ready to define our first brand of superstring theory – that of open R and NS strings with Neumann boundary conditions and GSO projected states. This is known as **Type I** superstring theory[14].

Aside from eliminating the tachyon, the GSO projection suffices to provide a spectrum with the same number of fermionic and bosonic degrees of freedom at each mass level. This is strong evidence for spacetime supersymmetry, which indeed one obtains manifestly by quantising the Green-Schwarz action equivalent to (2.11). The resulting spacetime supersymmetry for Type I string theory is $\mathcal{N} = 1$ in $D = 10$, motivating the nomenclature[15].

It only remains to consider the spectrum of closed superstrings. As observed in (2.21) and (2.34), they have twice the mode content of open strings, possessing independent left-moving (σ^+) and right-moving (σ^-) oscillations. Although the choice of ground state chirality in the Ramond sector for each

[14] Note that this necessarily also contains closed string states, by virtue of loops. On a more technical note, the open strings are in fact unoriented, and must possess $SO(32)$ Chan-Paton factors at each end for the cancellation of gauge anomalies. The calculation proving this fact launched the first superstring revolution.

[15] We presaged this above by calling the NS ground state a gauge boson, and the R ground state a gaugino.

set of modes is arbitrary, the relative sign between the modes is physically meaningful. This leads to two different brands of closed string theories.

In **Type IIA** string theory the left-moving and right-moving Ramond sector ground states are chosen to have opposite chiralities. After GSO projection the ground states are massless, and are given by

$$|-\rangle_{\mathrm{R}} \otimes |+\rangle_{\mathrm{R}}\,, \quad \tilde{b}^i_{-1/2}|0\rangle_{\mathrm{NS}} \otimes b^j_{-1/2}|0\rangle_{\mathrm{NS}}\,, \quad \tilde{b}^i_{-1/2}|0\rangle_{\mathrm{NS}} \otimes |+\rangle_{\mathrm{R}}\,, \quad |-\rangle_{\mathrm{R}} \otimes b^i_{-1/2}|0\rangle_{\mathrm{NS}} \tag{2.56}$$

In **Type IIB** string theory the left-moving and right-moving Ramond sector ground states are chosen to have the same chirality. After GSO projection the ground states are again massless, and obtained from (2.56) by taking $|-\rangle_{\mathrm{R}} \to |+\rangle_{\mathrm{R}}$. In the Green-Schwarz formalism one can show that the Type II string theories possess $\mathcal{N} = 2$ spacetime supersymmetry in $D = 10$.

Originally it was thought that the Type II string theories do not contain any open strings, since open strings possess a different amount of supersymmetry, so cannot couple consistently to the closed strings. However, we'll see in Section 3 that one may augment superstring theories with extended objects, to which open strings may couple, circumventing this argument.

We finish by summarising the massless field[16] content in each of the superstring theories we've encountered[17].

<u>Type I</u>

$\mathcal{N} = 1$ supergravity multiplet – graviton (35 NS-NS), two-form (28 NS-NS), dilaton (1 NS-NS), gravitino (56 NS-R), dilatino (8 NS-R).

$\mathcal{N} = 1$ super-Yang-Mills multiplet – vector (8 NS), gaugino (8 R).

<u>Type II</u>

$\mathcal{N} = 2$ supergravity multiplet – graviton (35 NS-NS), two-form (28 NS-NS), dilaton (1 NS-NS), gravitino (56 R-NS, 56 NS-R), dilatino (8 R-NS, 8 NS-R), vector (8 R-R, IIA only), three-form (56 R-R, IIA only), scalar (1 R-R, IIB only), two-form (28 R-R, IIB only), four-form (35 R-R, IIB only).

Note that in $D = 10$ the little group for massless representations is $SO(8)$, so it is most convenient to label the irreducible representations by their dimensions, rather than by Casimirs. By comparison the little group in $D = 4$ is $SO(2)$, so we label particles by the single Casimir, helicity.

[16]We have only first quantised the superstring, but in principle should view the excitations as fields ripe for second quantisation.

[17]In addition to the Type I, Type IIA and Type IIB theories, two other consistent superstring theories are known. These are the heterotic theories – closed string theories with $\mathcal{N} = 1$ spacetime supersymmetry where the right-movers come from 10-dimensional superstring theory and the left-movers come from 26-dimensional bosonic string theory. A full construction of such theories is beyond the scope of this review.

3 Duality

Much progress in string theory has been from the perspective of low energy effective actions. These naturally lead to identification of branes, natural objects on which gauge theories live. What's more, the effective approach provides a safe playground for the introduction of dualities. None is more influential than the AdS/CFT correspondence, which has become ubiquitous in modern theoretical physics. Many early career researchers encounter complicated brane constructions and holographic arguments, often without sufficient contextual information. Here, we collect some background arguments to fill the void.

3.1 Branes in M-Theory

Supergravity is a collection of massless theories with local supersymmetry and particle content with highest spin[18] 2. There are no supergravity theories with $D > 11$, since the smallest supermultiplet would necessarily contain higher spin fields. In fact, the $D = 11$ theory is unique up to normalization conventions, and hence known as **maximal supergravity**. The field content is relatively simple, viz.[19]

$$g_{\mu\nu}\ (\text{graviton, } 44)\,, \quad \Psi_\mu^a\ (\text{gravitino, } 128)\,, \quad C_{\mu\nu\rho}\ (\text{gauge field, } 84) \tag{3.1}$$

where we have explicitly stated the number of physical degrees of freedom for each field. The gauge transformations for $C_{(3)}$ are defined as[20]

$$C_{(3)} \to C_{(3)} + d\Lambda_{(2)} \tag{3.2}$$

for $\Lambda_{(2)}$ arbitrary. The requirements of gauge invariance, general coordinate covariance, local Lorentz invariance and local supersymmetry determine the action to be

$$16\pi G_{11} S = S_{\text{bosonic}} + S_{\text{fermionic}} \tag{3.3}$$

$$S_{\text{bosonic}} = \int d^{11}x\sqrt{-g}R - \frac{1}{2}\int F_{(4)} \wedge (*F)_{(7)} - \frac{1}{6}\int C_{(3)} \wedge F_{(4)} \wedge F_{(4)} \tag{3.4}$$

$$S_{\text{fermionic}} = \int d^{11}x \left[\frac{1}{2}\overline{\Psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \Psi_\rho - \frac{1}{192}\left(\overline{\Psi}_\mu \Gamma^{\mu\nu\rho\lambda\sigma\tau}\Psi_\tau + 12\overline{\Psi}^\nu \Gamma^{\rho\lambda}\Psi^\sigma\right)\left(F + \hat{F}\right)_{\nu\rho\lambda\sigma}\right] \tag{3.5}$$

where G_{11} is Newton's constant, R is the Ricci scalar, $F_{(4)} = dC_{(3)}$ is the gauge field strength, D_ν is the gravity covariant derivative, $\Gamma^{\mu\dots\nu}$ is the antisymmetrised product of gamma matrices, and $\hat{F}_{(4)}$ is the supercovariant gauge field strength[21]. The bosonic action comprises an Einstein-Hilbert term, a kinetic term and a **Chern-Simons term**[22].

[18] The definition of spin is subtle in $D = 11$. In particular, a generic massless representation of the little group is labelled by the eigenvalue of more than one Casimir. Colloquially, we say that a symmetric spinor representation with n indices has spin $n/2$. To be rigorous, we require a supergravity theory to possess no higher spin fields after compactification on a torus to $D = 4$.

[19] We employ spacetime indices $\mu, \nu, \dots = 0, \dots 10$ and Majorana spinor indices $a, b, \dots = 1, \dots 32$.

[20] When referring to forms in index-free notation, we use a bracketed subscript to denote the rank of the form.

[21] A supercovariant quantity is one whose SUSY transformation does not contain any derivatives of the SUSY parameter. We can construct such a quantity from $F_{(4)}$ by adding a term quadratic in gravitinos.

[22] An action built from Chern-Simons terms would lead to correlation functions with no metric dependence, hence only sensitive to topological data. The converse is not true – there exist topological QFTs not built from Chern-Simons terms.

The first port-of-call when considering any new theory is to determine its classical solutions. To make this problem tractable, we consider only the bosonic part of the action. In fact, one can show that this is a **consistent truncation**, in the sense that all solutions of the truncated theory are also solutions of the full theory. To further simplify matters, we seek solutions with Poincaré invariance in $p+1$ dimensions and $SO(11-p-1)$ invariance in the transverse dimensions. The simplest metric ansatz respecting such a symmetry is

$$ds^2 = e^{2\alpha(r)} dx^M dx^N \eta_{MN} + e^{2\beta(r)} dx^m dx^n \delta_{mn} \tag{3.6}$$

where r is the radial coordinate in the transverse space. We denote worldvolume[23] coordinates by capital letters $M, N = 0, \ldots p$ and transverse coordinate by lowercase letters $m, n = p+1, \ldots 11$. For the gauge field there are two obvious choices. First, assume the gauge field couples only to the worldvolume, in which case $p = 2$ and we may choose

$$C_{MNR} = \epsilon_{MNR}\gamma(r)\,, \quad F_{MNRa} = \epsilon_{MNR}\partial_a\gamma(r) \tag{3.7}$$

Second, observe that the bosonic action has an **electric-magnetic dual**[24] obtained by sending $F_{(4)} \to (*F)_{(7)}$ where $*$ is the Hodge star. So the analogue of (3.7) in the dual theory has $p = 5$ and

$$\tilde{C}_{MNRST} = \tilde{\gamma}(r)\epsilon_{MNRST} \tag{3.8}$$

We can use the duality mapping to express $\tilde{C}$ in terms of the original theory, whence we find a field strength coupling only to the transverse directions

$$F_{abcd} = (\sqrt{-g})^{-1}\epsilon_{abcde}\partial^e\tilde{\gamma}(r) \tag{3.9}$$

We refer to (3.7) as the electric solution and (3.9) as the magnetic solution, since the former has nonzero conserved charge

$$e = \int_{\partial\mathcal{M}_8}\left((*F)_{(7)} + \frac{1}{2}C_{(3)} \wedge F_{(4)}\right) \tag{3.10}$$

analogously to a Maxwell electic monopole, while the latter has nonzero conserved charge

$$\mu = \int_{\partial\tilde{\mathcal{M}}_5} F_{(4)} \tag{3.11}$$

analogously to a Maxwell magnetic monopole, where $\mathcal{M}_8$ is parameterised by the transverse coordinates in the original theory and $\tilde{\mathcal{M}}_5$ is parameterised by the transverse coordinates in the dual theory.

For appropriate choices of $\alpha(r), \beta(r), \gamma(r), \tilde{\gamma}(r)$ one can prove that the ansätze not only satisfy the bosonic supergravity equations of motion, but are also invariant under half the supersymmetry. This implies[25] that they satify a **BPS bound** – after quantization they represent the lightest states with

[23] The reason for this nomenclature will become apparent when we define p-branes later.

[24] A dual theory is a mathematically distinct yet physically equivalent description of reality.

[25] See any introductory text on supersymmetry for the link between supersymmetric and BPS solutions.

given quantum numbers. As such we expect these solutions to yield stable quantum-mechanical objects, as decay is energetically forbidden.

We have thus constructed two stable supersymmetric extended solutions of supergravity. We might hope to view these as the low-energy, classical approximations of higher-dimensional generalisations of the superstring. With this interpretation in mind, we define a general **p-brane** to be a quantum extended object obtained from some supersymmetric generalisation of the Dirac action[26]

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det\left(h_{\alpha\beta}\right)} \tag{3.12}$$

where $h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ is the induced metric. With this definition, it can be shown that the electric and dual magnetic solutions above emerge from the low energy limit of M2- and M5-branes. The putative quantum theory with M2- and M5-branes as fundamental objects was given the moniker **M-theory** by Edward Witten, where M is deliberately ambiguous, hence the names. The supergravity action (3.3) is expected to be the low energy effective action of M-theory.

One could imagine having Mp-branes for $p \neq 2, 5$ also. However, these objects are likely to be unstable[27]. Firstly, they don't correspond to BPS solutions of supergravity. Secondly, generalising our experience with M2- and M5- branes, we expect that an Mp-brane will couple to a $(p+1)$-form gauge theory in spacetime. But such fields don't appear in the maximal supergravity action unless $p = 2$ or $p = 5$.

It is worth emphasising that one can view p-branes from two perspectives. For aficionados of supergravity, branes are solitonic[28] solutions of the equations of motion acting as sources for gauge fields and curvature in spacetime. Often the resulting spacetime is geodesically incomplete, with properties analogous to a Reissner-Nordström black hole, hence the term **black brane**[29]. From a M-theory perspective, a p-brane is a dynamical object in its own right, coupling to the background fields of supergravity in a low-energy probe[30] approximation.

3.2 Branes in Superstring Theory

Supergravity theories in 10 dimensions come in three types, according to the variety of supersymmetry they possess. These are referred to as Type I, Type IIA and Type IIB supergravity, prefiguring a connection to superstrings. They possess $(1,0)$, $(1,1)$ and $(2,0)$ supersymmetry respectively, where we have

[26] In the next section, we'll encounter D-branes supporting gauge fields on their worldvolume. To incorporate this, one generalizes (3.12) by including a gauge field strength $F_{\alpha\beta}$ in the deteminant, defining the DBI action.

[27] The arguments for instability aren't completely watertight – M-theory also possesses stable W0, KK7 and KK9 branes, where W stands for wave and KK for Kaluza-Klein.

[28] A soliton is a particle-like solution, in the sense that the non-zero field values are localized in spacetime, though not necessarily to 0 dimensions.

[29] The electric BPS brane above is black, but remarkably its magnetic dual is not!

[30] A probe brane is one which has no backreaction on the background fields.

explicitly indicated the chirality of the chosen generators[31].

A simple way of deriving the field content and Lagrangian for Type IIA supergravity is by performing a **dimensional reduction** of 11-dimensional maximal supergravity. The procedure is as follows: take the x^{10} dimension to be a circle of radius R, Fourier expand the fields on the circle and discard the non-zero modes. This is to be contrasted with **Kaluza-Klein compactification** in which all modes are kept.

We illustrate the procedure in the bosonic sector. The 11-dimensional metric takes the form[32]

$$G_{MN} = e^{-2\phi/3}\begin{pmatrix} g_{\mu\nu} + e^{2\phi}A_\mu A_\nu & e^{2\phi}A_\mu \\ e^{2\phi}A_\nu & e^{2\phi} \end{pmatrix} \tag{3.13}$$

and so contains a scalar dilaton field ϕ, 10-dimensional metric g, and abelian vector gauge field $A_{(1)}$ with corresponding field strength $F_{(2)}$. Meanwhile, the 11-dimensional 3-form yields

$$C^{(11)}_{\mu\nu\rho} = C_{\mu\nu\rho}\,, \quad C^{(11)}_{\mu\nu 11} = B_{\mu\nu} \tag{3.14}$$

namely a 3-form field $C_{(3)}$ and a 2-form field $B_{(2)}$, with corresponding field strengths $F_{(4)}$ and $H_{(3)}$. After substituting these definitions in (3.4) and integrating out the circular coordinate x^{10} we obtain the bosonic part of the Type IIA supergravity action

$$16\pi G_{10} S_{\text{bosonic}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}} \tag{3.15}$$

$$S_{\text{NS}} = \int d^{10}x\; e^{-2\phi}\sqrt{-g}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_{(3)}|^2\right) \tag{3.16}$$

$$S_{\text{R}} = -\frac{1}{2}\int d^{10}x\sqrt{-g}\left(|F_{(2)}|^2 + |\tilde{F}_{(4)}|^2\right) \tag{3.17}$$

$$S_{\text{CS}} = -\frac{1}{2}\int B_{(2)}\wedge F_{(4)}\wedge F_{(4)} \tag{3.18}$$

where $\tilde{F}_{(4)} = dC_{(3)} - A_{(1)}\wedge H_{(3)}$. We have suggestively split the action above into Ramond, Neveu-Schwarz and Chern-Simons pieces. Indeed we may identify the dynamical fields $\phi, g, B_{(2)}$ appearing in S_{NS} with the NS-NS sector dilaton, graviton and two-form in the massless spectrum of Type II superstrings. Similarly S_{R} contains the vector $A_{(1)}$ and three-form $C_{(3)}$ comprising the massless states in the R-R sector of Type IIA superstrings. In fact one can prove that in the low energy $\alpha' \to 0$ limit, the effective action of Type IIA superstring theory is precisely Type IIA supergravity[33].

Moreover, by a short series of calculations relating G_{10}, G_{11}, g_s, α' and 11-dimensional Planck length,

[31] In $D = 2 \pmod 4$ representations of the Lorentz algebra admit spinors which are simultaneously Majorana and Weyl, so one may define chiral supersymmetry generators.

[32] We use indices $M, N = 0, 1, \ldots 10$ and $\mu, \nu = 0, 1, \ldots 9$.

[33] To derive this, start by expanding around a constant classical field configuration à la $X^\mu(\sigma,\tau) = \bar{x}^\mu + \sqrt{\alpha'}Y^\mu(\sigma,\tau)$, where α' appears for dimensional reasons. Strictly speaking the dimensionless expansion parameter should be $\sqrt{\alpha'}/r$ where r is the radius of curvature of the background spacetime.

we may derive that the radius of the x^{10} circle determines the string coupling via

$$g_s = \frac{R}{\sqrt{\alpha'}} \tag{3.19}$$

We might therefore suppose that M-theory reduces to Type IIA superstring theory upon applying dimensional reduction. This conjecture allows us to instantly determine the stable p-branes. Indeed depending on whether an M2- or M5-brane wraps the compact direction, we obtain p-branes with $p = 1, 2, 4, 5$. We may catalogue these objects by considering the background form fields to which they couple.

In the previous section we saw that a p-brane naturally couples to an $(p+1)$-form field, just as a particle couples to the electromagnetic field in $D = 4$. We immediately deduce that the brane with $p = 1$ couples to the NS-NS two-form $B_{(2)}$, motivating the terminology **NS1-brane**. Of course, this object is already familiar – it is nothing but the fundamental superstring! Clearly the R-R and NS-NS sectors are preserved under electric-magnetic duality, so we identify the $p = 5$ case as an **NS5-brane**. Similarly the $p = 2$ and $p = 4$ branes are electric-magnetic duals, and they couple to the R-R sector form fields $C_{(3)}$ and $(*C)_{(5)}$ respectively. We refer to them as **D2-** and **D4-branes**.

Pushing the coupling paradigm further, we are led to define branes coupling to the R-R sector fields $A_{(1)}$ and $(*A)_{(7)}$. According to the nomenclature defined above, we should refer to these as **D0-** and **D6-branes**. Their M-theory origin is from the slightly obscure W0 and KK7 branes mentioned in footnote 27. By this logic we should also gain a **D8-brane** from the M-theory KK9-brane. This couples to an RR 9-form which may be consistently included in Type IIA supergravity, defining Romans massive supergravity[34]. The corresponding 10-form field strength does not have any propagating degrees of freedom but does carry energy density, so is appealing for cosmological constant phenomenology.

It is time to explain our mysterious decision to label the branes coupling to Ramond-Ramond fields by the letter D. We define a **Dp-brane** to be the object on which an open string ends when equipped with $p + 1$ Dirichlet boundary conditions (2.17). In Section 2.2 we argued that such objects must be dynamical, because momentum is transferred off the end of the open string. In fact, we may identify the stable D-branes as exactly the objects which couple to RR-sector background fields.

Polchinski's famous argument[35] goes as follows. Let us calculate the scattering amplitude A between two Dp-branes. By definition, this is given at leading order in g_s by a 1-loop vacuum graph of open strings, which may be explicitly evaluated. But **open-closed string duality** tells us that A is equivalently determined by the exchange of a single closed string between the branes. Moreover, there is a well-defined prescription for extracting from A the contribution from the RR-sector of closed strings.

[34]There is no known direct derivation of Romans supergravity from 11-dimensional supergravity.
[35]To a great extent this kickstarted the second superstring revolution of the mid-1990s.

Taking a low-energy limit, we find

$$-V_{p+1}2\pi(4\pi^2\alpha')^{3-p}G_{9-p} \tag{3.20}$$

where V_{p+1} is the volume of the brane and G_{9-p} is the scalar propagator in $9-p$ dimensions. Amazingly, this is also the amplitude for the exchange of a Ramond-Ramond $(p+1)$-form field between the branes, proving that D-branes must naturally couple to the RR-sector.

The dynamics of a D-brane[36] are determined by the excitations of open strings with the Dirichlet boundary conditions. More explicitly, the massless bosonic spectrum of an open string with Dirichlet conditions for $a = 0, \dots p$ and Neumann for $I = p+1, \dots D$ contains scalar fields ϕ^I governing the transverse oscillation of the brane, and an abelian gauge field A^a living on the worldvolume of the brane. We may construct a $U(N)$ non-abelian gauge theory as the field theory carried by a **stack** of N coincident D-branes. This observation underpins both braneworld phenomenology and the AdS/CFT correspondence described in Section 3.3.

The open string definition of Dp-branes provides another derivation of the values of p for which they are stable. Recall that we must perform a GSO projection to remove tachyons from superstring theory. Given a choice of closed string projection (Type IIA or Type IIB) we may obtain a consistency condition on open string projection via open-closed string duality. This restricts the possible values of p for which a Dp-brane is stable to p even for Type IIA and p odd for Type IIB superstring theory.

At first glance, the difference in GSO projection and D-brane stability between the two Type II string theories suggests that they are completely separate descriptions of physics. However, these mathematical distinctions mask the fact that the theories become equivalent upon compactification. More explicitly, the Type IIA theory compactified on a circle of radius R is exactly the Type IIB theory compactified on a circle of radius α'/R. This equivalence is known as **T-duality**.

T-duality is a perturbative duality, in that the string coupling transforms as $g_s \to g_s\sqrt{\alpha'}/R$. In particular it is valid in the leading (tree or classical) approximation. Furthermore, as an exercise, the reader may verify that the effect of T-duality in the X^9 direction is to map

$$X_L^9 \to X_L^9\,, \quad X_R^9 \to -X_R^9\,, \quad \psi_+^9 \to \psi_+^9\,, \quad \psi_-^9 \to -\psi_-^9 \tag{3.21}$$

By virtue of (2.25) we see that T-duality exchanges Dirichlet and Neumann boundary conditions for the X^9 coordinate, explaining why the dimensionality of stable branes differs by 1 between Type IIA and

[36] The fact that we can determine the low-energy degrees of freedom of a D-brane makes them valuable for model-building. By contrast much less is understood about NS-branes, so they appear more rarely in constructions at the time of writing. In M-theory, the worldvolume theory of coincident M2-branes is known to be a Chern-Simons theory called ABJM, but little is known about the M5 case.

Type IIB superstring theory.

The dualities between and within string theories we have encountered in this section are by no means an exhaustive list. Principal among the omissions is **S-duality**, so named because it is a strong-weak duality $g_s \to 1/g_s$. In fact, one can often embed S-duality and T-duality into a larger space of duality symmetries called **U-duality**. Sadly, such topics are beyond the scope of this course, and we instead refer the reader to any modern textbook covering advanced string theory.

3.3 AdS/CFT

The AdS/CFT correspondence relates operators and their expectation values in two distinct theories[37]. Typically a string theory in a (negatively curved) spacetime is dual to a (conformal) field theory in one fewer dimensions, which may be viewed as living on the boundary[38]. For this reason, the correspondence is a realisation of the **holographic** principle, whereby boundary dynamics suffice to reconstruct bulk physics. The most well-developed AdS/CFT correspondence involves bulk Type IIB strings in an $AdS_5 \times S^5$ background and boundary $\mathcal{N} = 4$ super-Yang-Mills theory. We shall focus on this example henceforth.

The first fundamental claim of AdS/CFT is that there is a bijection

$$\text{bulk string states } V_i \longleftrightarrow \text{boundary operators } \mathcal{O}_i \tag{3.22}$$

such that the quantum numbers match. Indeed one can show that the global symmetries of $\mathcal{N} = 4$ correspond to the gauge symmetries of IIB strings in AdS_5, making this matching plausible.

More concretely, one can uniquely identify the AdS duals to single trace operators in the CFT by matching representations of the superconformal group, indexed by quantum scaling dimension Δ, chiral spin (j_1, j_2), $SU(4)$ R-symmetry representation and supersymmetry (if any). For example, in $\mathcal{N} = 4$ there exists a $\frac{1}{2}$-BPS supermultiplet, by definition annihilited by half of the supercharges. This is the **stress-tensor multiplet**, containing among other fields

$$\mathrm{Tr}(\phi_i \phi_j) - \frac{\delta_{ij}}{6}\mathrm{Tr}(\phi_k \phi_k) \quad \text{and} \quad T_{\mu\nu} \tag{3.23}$$

where ϕ_i for $i = 1, \ldots 6$ are real scalar fields. In supergravity the corresponding $\frac{1}{2}$-BPS multiplet contains

$$\Phi_{ij} \quad \text{and} \quad g_{\mu\nu} \tag{3.24}$$

whence we may immediately identify fields based on their tensor structure. The fact that the stress-tensor T corresponds to the graviton g is a quite generic feature of holography, so gravity is always

[37] Anti-de-Sitter space (AdS) is the maximally symmetry solution to Einstein's equations with negative cosmological constant. It is the Lorentzian analogue of hyperbolic space.

[38] The bracketed terms are common conditions, but are not thought to be necessary.

present on one side of the duality[39].

The second fundamental claim of AdS/CFT is that there is an equality

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\text{CFT}} = \langle V_1 \cdots V_n \rangle_{\text{string}} \tag{3.25}$$

between the correlation functions of gauge-invariant local operators $\mathcal{O}_i$ in the CFT, and the S-matrix elements of corresponding open string states represented by vertex operators V_i in the string theory. To make this statement precise, we must relate the bulk and boundary parameters, defining

$$g_s = g_{\text{YM}}^2\,, \quad L^4/\alpha'^2 = 4\pi\lambda \tag{3.26}$$

where L is the radius of AdS_5, the boundary gauge group is $SU(N)$ and $\lambda = g_{\text{YM}}^2 N$ is the 't Hooft coupling.

It's time to examine a motivational example. On the CFT side, define the Maldecena Wilson loop[40]

$$W(\mathcal{C}) = \frac{1}{N} \mathrm{Tr}\,\mathcal{P} \exp\left[i \oint_{\mathcal{C}} ds \left(A_\mu(x^\mu(s))\dot{x}^\mu + \theta^I(s)\Phi^I(x^\mu)\sqrt{\dot{x}^2} \right) \right] \tag{3.27}$$

where $\mathcal{C} = \{x^\mu(s)\}$ is a contour on $\mathbb{R}^4$ and $\theta^I(s)$ is a map taking each point of the loop to a point on S^5. According to (3.22), this is dual to an open string ending on the contour $\{x^\mu(s), \theta^I(s)\}$ on the boundary of $AdS_5 \times S^5$. At first glance, it seems we have pulled this identification out of thin air. In fact, this follows quite transparently from the original construction promoting AdS/CFT, as follows.

Consider a stack of N $D3$-branes in flat space. The worldvolume theory of the branes is precisely $\mathcal{N} = 4$ super-Yang-Mills theory, with $SU(N)$ gauge group. Open strings ending on the branes couple to the gauge field A^μ parallel to the branes and scalar degrees of freedom Φ^I describing the transverse motion of the branes. Of course, we may calculate the correlation function of a Wilson loop involving A^μ and Φ^I purely within the four-dimensional field theory. But we may perform an equivalent calculation from the the perspective of open strings in an $AdS_5 \times S^5$ geometry produced from the backreaction of the branes on the bulk spacetime[41]. The equality of the answers is precisely the statement (3.25).

Unfortunately, it is unfeasible to explicitly test (3.25), since it is not known how to consistently quantise strings in an $AdS_5 \times S^5$ background. To make progress, we must therefore judiciously choose the parameters in (3.26) such that the string theory enters a calculable regime. Two of the most commonly considered limits are

$$\text{planar CFT},\ N \to \infty\,,\ \lambda \text{ fixed} \quad \longleftrightarrow \quad \text{classical string theory},\ g_s \to 0 \tag{3.28}$$

[39] Recall that a massless spin 2 particle is necessarily a graviton, as famously proved by Weinberg in 1964.

[40] This is essentially just a supersymmetrisation of the ordinary Wilson loop.

[41] It is tempting to think of the stack of branes as living at the boundary of AdS_5. Nothing could be further from the truth! The branes are responsible for warping the spacetime, so in fact they lie deep within the bulk. Indeed, the holographic nature of AdS/CFT is not immediately manifest from this brane construction.

$$\text{strongly coupled CFT, } \lambda \to \infty \quad \longleftrightarrow \quad \text{quantum type IIB supergravity, } \alpha' \to 0 \tag{3.29}$$

We may now verify the claim (3.25) for the Wilson loop (3.27). In the limit (3.28) the claim becomes (courtesy of a saddle-point approximation)

$$\langle W(\mathcal{C})\rangle_{\text{CFT}} = e^{-S_{\text{on-shell}}} \tag{3.30}$$

where $S_{\text{on-shell}}$ is the string action evaluated on the classical worldsheet solution ending on $\{x^\mu(s), \theta^I(s)\}$. Of course, this is nothing other than the minimal area of a surface with the specified boundary values. In the **Poincaré patch** the AdS metric is

$$ds^2 = \frac{L^2}{z^2}\left(dx^\mu dx_\mu + dz^2\right) \tag{3.31}$$

for radial coordinate $0 < z < \infty$. This diverges at the boundary $z = 0$, so generically the minimal area surface drops down into the bulk. Moreover, this divergence implies that the area must be regularised by applying some cutoff in z for the right-hand-side of (3.30) to be meaningful.

To perform an explicit calculation, we must specify a contour $\mathcal{C}$. The most obvious choice is a rectangle embedded in the x^0–x^1 plane, with spacelike separation L and timelike separation $T \gg L$. It is a foundational fact of lattice QCD that a Wilson loop on such a contour computes the effective potential $V_{\text{eff}}(L)$ between a static quark and antiquark viewed as quantum mechanical particles via[42]

$$\langle W(\square)\rangle_{\text{CFT}} = e^{-TV_{\text{eff}}(L)} \tag{3.32}$$

To verify the AdS/CFT correspondence, we will focus only on the power law behaviour of the effective potential. In any conformal field theory, the absence of dimensionful parameters implies that $V_{\text{eff}}(L) \sim L^{-1}$. On the AdS side we may calculate the regularised area in the limit (3.29) via background field approximation, and obtain exactly $V_{\text{eff}}(L) \sim L^{-1}$ as required[43,44].

The **GKPW formula** provides an explicit realisation of (3.25) for general correlators in the limit (3.29). Let $h(x, z)$ denote a supergravity field in the bulk, with boundary values $g(x, 0)$. Let $\mathcal{O}$ be the corresponding operator in the CFT. Then

$$Z_{\mathcal{O},\text{CFT}}\left[g(x)\right] = \int_{g(x)} \mathcal{D}\phi \; e^{-S[\phi]} \tag{3.33}$$

[42]The full derivation of this result is due to Barchielli, Montaldi and Prosperi, and is quite involved. One starts with the gauge invariant operator representing a quark-antiquark bound state $\mathcal{O}(t) = \bar{\psi}(t,0)U(t,0,t,L)\psi(t,L)|0\rangle$, where U is a Wilson line connecting $(t, 0)$ and (t, L). The amplitude for this state at time 0 to propagate into a similar state at time T is the overlap $\langle\mathcal{O}(0)\mathcal{O}(T)\rangle$. Path integrating out the quarks, one finds that the correlator satisfies a Schrödinger-like equation, admitting the interpretation (3.32).

[43]The minimal area result depends on the 't Hooft coupling as $\sqrt{\lambda}$, at odds with the perturbative CFT prediction of λ^1. This illustrates that the background field approximation breaks down before one enters the perturbative region of $\mathcal{N} = 4$ super-Yang-Mills.

[44]In flat space, the minimal surface would have stayed at the boundary, leading to a confining potential $V_{\text{eff}}(L) \sim L$.

where ϕ denotes all supergravity fields, $S[\phi]$ is the Type IIB supergravity action, $\int \mathcal{D}h$ is subject to the boundary condition $h(x,0) = g(x)$ and

$$Z_{\mathcal{O},\mathrm{CFT}}\left[J\right] = \int \mathcal{D}\psi \exp\left(\int d^4x \ \mathcal{L}[\psi] + \mathcal{O}(x)J(x)\right) \tag{3.34}$$

is the generating functional for correlators of the CFT, and ψ denotes all CFT fields. Generically it is not known how to compute the path integral in (3.33). For practical purposes, one may simultanously pass to the limit (3.28), in which case the GKPW formula reduces to

$$Z_{\mathcal{O},\mathrm{CFT}}\left[g(x)\right] = e^{-S_{\text{on-shell}}[\phi]} \tag{3.35}$$

Alternatively, one may evaluate the supergravity partition function explicitly by determining supergravity propagators and calculating the resulting Feynman diagrams, which are known as **Witten diagrams**. Sample computations can be found in more extensive introductions to AdS/CFT.

4 Ambitwistor Strings

Methods and inspiration from string theory have played an increasingly important role in the study of field theory over the past 30 years. Here we review a recent application, revealing surprisingly compact formulae for tree-level amplitudes. This section serves the purpose of exposing the student to the language and notation of contemporary amplitudes, assumed knowledge for a growing number of talks.

4.1 MS Action

The ambitwistor string is a chiral infinite-tension version of the RNS string we introduced in Section 2.1, living inside the space of complexified null geodesics, also known as **ambitwistor space**. The scattering amplitudes of the ambitwistor string yield particularly compact forms of field theory amplitudes in various theories, depending on the matter content on the worldsheet.

To introduce the ambitwistor string, first recall that the **phase space action** yielding Hamilton's equations may be written

$$S[x,p] = \int p\ dx - H(x,p)\ dt \tag{4.1}$$

For example, we may write the phase space version of the worldine action (2.1) for a massless particle as

$$S[x^\mu, p_\mu] = \int p_\mu dx^\mu - \frac{e}{2} p_\mu p^\mu d\tau \tag{4.2}$$

by defining the conjugate momentum[45]

$$p_\mu = e^{-1}\dot{x}_\mu \tag{4.3}$$

In this formalism the gauge transformations (2.2) become

$$\delta_\xi X^\mu = \xi P^\mu\,, \quad \delta_\xi P_\mu = 0\,, \quad \delta_\xi e = d\xi \tag{4.4}$$

and e is easily seen to be a Lagrange multiplier enforcing the null condition $p^2 = 0$.

From (4.2) we may obtain the bosonic ambitwistor string action by complexifying both the worldsheet and the target space. Moreover, we require the model to be chiral, with the action involving only derivatives $\bar{d} \equiv d\bar{z}\ \partial_{\bar{z}}$ in harmony with Witten's original twistor string construction. Therefore we arrive at the **bosonic MS action**[46]

$$S[X^\mu, P_\mu] = \frac{1}{2\pi}\int P_\mu \bar{\partial} X^\mu - \frac{e}{2} P_\mu P^\mu \tag{4.5}$$

where we interpret X^μ as a map from the worldsheet to ambitwistor space, P_μ as a $(1,0)$-form, and e as a tangent-bundle-valued $(0,1)$-form[47].

[45] Here and in the string case the conjugate momentum coincides with the total momentum, since the theories are free.

[46] We use the Dolbeault operators $\partial = dz\partial_z$ and $\bar{\partial} = d\bar{z}\partial_{\bar{z}}$.

[47] For concreteness, recall that a (p,q) form may be written $\omega = f(z,\bar{z})(dz)^p(d\bar{z})^q$ in some coordinates $(z,\bar{z})$ on the worldsheet, where f is an arbitrary (not necessarily holomorphic or anti-holomorphic) smooth function.

Alternatively we may exhibit (4.5) as a degeneration of the bosonic string (2.3) by taking the $\alpha' \to 0$ limit in a chiral manner. We first expand the auxiliary inverse metric h in terms of scalar fields $(\Omega, e, \bar{e})$ writing

$$h^{\alpha\beta}\partial_\alpha\partial_\beta = \Omega\left(\partial_z\partial_{\bar{z}} + e\partial_z\partial_z + \bar{e}\partial_{\bar{z}}\partial_{\bar{z}}\right) \tag{4.6}$$

so that the Polyakov action becomes, after a Weyl rescaling,

$$S = \frac{1}{2\pi\alpha'}\int dz d\bar{z}\frac{1}{\sqrt{1-e\bar{e}}}\left(\partial X^\mu \bar{\partial} X_\mu + e\partial X^\mu \partial X_\mu + \bar{e}\bar{\partial} X^\mu \bar{\partial} X_\mu\right) \tag{4.7}$$

It is an elementary exercise to show this is equivalent to the action

$$S = \frac{1}{2\pi\alpha'}\int dz d\bar{z}\frac{1}{\sqrt{1-e\bar{e}}}\left(P^\mu \bar{\partial} X_\mu + \bar{P}^\mu \partial X_\mu - P^\mu \bar{P}_\mu + eP^\mu P_\mu + \bar{e}\bar{P}^\mu \bar{P}_\mu\right) \tag{4.8}$$

where we have introduced additional non-dynamical fields P^μ and $\bar{P}^\mu$. Furthermore, we are free to redefine our auxiliary fields, allowing us to introduce a preferred chirality via

$$P \to \alpha' P\,, \quad \bar{P} \to \alpha'^2 \bar{P}\,, \quad e \to \alpha'^{-1} e\,, \quad \bar{e} \to \alpha'^2 \bar{e} \tag{4.9}$$

The action becomes

$$S = \frac{1}{2\pi}\int dz d\bar{z}\frac{1}{\sqrt{1-\alpha' e\bar{e}}}\left(P_\mu \bar{\partial} X^\mu + eP_\mu P^\mu + \alpha' \bar{P}_\mu \partial X^\mu + \alpha' \bar{e}\bar{P}_\mu \bar{P}^\mu - \alpha'^2 P_\mu \bar{P}^\mu\right) \tag{4.10}$$

yielding (4.5) in the limit as $\alpha' \to 0$. From this perspective it is not surprising the the critical dimension for the supersymmetric ambitwistor string is 10.

Counterintuitively, we will now specialize to the case $D = 4$, where quantum anomalies render the theory inconsistent[48]. Nevertheless, if we remain at tree level, ambitwistor string scattering amplitudes yield correct and compact expressions for field theory amplitudes. Indeed $D = 4$ is an ideal mathematical playground in which to explore ambitwistor strings, since then ambitwistor space has a convenient parameterisation as a quadric inside $\mathbb{PT} \times \mathbb{PT}^*$, a Cartesian product of **projective twistor spaces**.

More explicitly[49] given a null geodesic through x with momentum[50] $\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ we define a twistor $Z^a \in \mathbb{T} = \mathbb{C}^4$ and a dual twistor $W_a \in \mathbb{T}^* = \mathbb{C}^4$ via the **incidence relations**

$$Z^a = (\tilde{\lambda}_{\dot{\alpha}}, \tilde{\mu}^\alpha) = (\tilde{\lambda}_{\dot{\alpha}}, -ix^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}})\,, \quad W_a = (\mu^{\dot{\alpha}}, \lambda_\alpha) = (ix^{\alpha\dot{\alpha}}\lambda_\alpha, \lambda_\alpha) \tag{4.11}$$

[48] It is possible that one could add further matter content to the theory and reduce the critical dimension to $D = 4$ for loop-level consistency. This remains an open problem.

[49] Henceforth we use Greek indices $(\alpha, \beta, \dots)$ to denote spinor-helicity components on spacetime, and complex coordinates $(z, \bar{z})$ on the worldsheet.

[50] We can express an arbitrary vector x^μ in spinor-helicity language using a 4-vector of Pauli matrices, viz. $x^{\alpha\dot{\alpha}} = x^\mu \sigma_\mu^{\alpha\dot{\alpha}}$. This is convenient because null vectors p^μ then naturally yield simple tensors $\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$, by virtue of the relation (exercise) $\det(p^{\alpha\dot{\alpha}}) = p^\mu p_\mu$. The spinors λ and $\tilde{\lambda}$ are the most basic ingredients of scattering amplitudes.

Clearly (Z, W) corresponds to a null geodesic iff $Z^a W_a = 0$. More generally, we see that twistor space and Minkowski space are related by a point-line duality

$$\begin{aligned} \text{point } x_i \quad &\longleftrightarrow \quad \text{line through } Z_{i-1} \text{ and } Z_i \,, \\ \text{null line through } x_i \text{ and } x_{i+1} \quad &\longleftrightarrow \quad \text{point } Z_i \,. \end{aligned} \tag{4.12}$$

such that the light-cone (and therefore conformal) structure of the theory is trivialised in twistor space. This observation was a central tenet of the original twistor programme of Penrose, who argued that the divergences of quantum field theory could be tamed by performing quantisation instead in the non-local setting of twistor space.

We now write the action (4.5) in terms of (Z, W). It is an easy exercise to show that

$$P_\mu dX^\mu = \frac{i}{2}\left(Z^a dW_a - W_a dZ^a\right) \tag{4.13}$$

using the incidence relations (4.11). Therefore

$$S = \frac{1}{2\pi}\int W_a \bar{\partial} Z^a - Z^a \bar{\partial} W_a + a Z^a W_a \tag{4.14}$$

imposing chirality as above, where we interpret W and Z as $\left(\frac{1}{2}, 0\right)$-forms and a as a $(0,1)$-form.

4.2 Vertex Operators

To calculate string scattering amplitudes we require vertex operators. By the arguments of Section 1.2, these are the first-quantised wavefunctions for external states, reinterpreted as worldsheet operator insertions. Of course, we require these to be transformed from spacetime to ambitwistor space. Our strategy will be to determine the wavefunctions on (dual) twistor space, yielding appropriate vertex operators by a pullback to the quadric $Z^a W_a = 0$.

The **massless field equations** on spacetime may be written in spinor notation as

$$\partial_{\alpha_1 \dot{\alpha}_1} \phi^{\alpha_1 \ldots \alpha_h}(x) = 0 \quad \text{for helicity } +h/2 \tag{4.15}$$

$$\partial_{\alpha_1 \dot{\alpha}_1} \phi^{\dot{\alpha}_1 \ldots \dot{\alpha}_h}(x) = 0 \quad \text{for helicity } -h/2 \tag{4.16}$$

$$\partial_{\alpha \dot{\alpha}} \partial^{\alpha \dot{\alpha}} \phi(x) = 0 \quad \text{for helicity } 0 \tag{4.17}$$

where ϕ is symmetric in all its indices and $h \in \mathbb{Z}_{>0}$. Fourier transforming immediately reveals the solutions of definite momentum $p_a^{\alpha\dot{\alpha}} = \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}}$ to be[51]

$$\phi_a^{\alpha_1 \ldots \alpha_h}(x) = \lambda_a^{\alpha_1} \ldots \lambda_a^{\alpha_h} e^{ip\cdot x} \,, \quad \phi_a^{\dot{\alpha}_1 \ldots \dot{\alpha}_h}(x) = \tilde{\lambda}_a^{\dot{\alpha}_1} \ldots \tilde{\lambda}_a^{\dot{\alpha}_h} e^{ip\cdot x} \tag{4.18}$$

By the same coincidence that led to the mysterious success of second quantisation in the early days of quantum field theory, these solutions are precisely the position space wavefunctions for momentum

[51] Here a is simply a label, rather than the index of a twistor.

eigenstates we require.

The **Penrose transform** relates solutions to the massless field equations of helicity $h/2$ to holomorphic objects of homogeneity $h-2$ on projective twistor space. More rigorously, we must consider cohomology classes $\{\bar{\partial} f = 0\}/\{f = \bar{\partial} g\}$ of smooth $(0,1)$-forms f. We may then write

$$\phi^{\alpha_1 \dots \alpha_h}(x) = \frac{1}{2\pi i} \int_{\mathbb{CP}^1} \lambda^{\alpha_1} \dots \lambda^{\alpha_h} f(ix^{\beta\dot{\beta}}\lambda_\beta, \lambda_\beta) \wedge \lambda^\gamma d\lambda_\gamma \tag{4.19}$$

$$\phi^{\dot{\alpha}_1 \dots \dot{\alpha}_h}(x) = \frac{1}{2\pi i} \int_{\mathbb{CP}^1} \frac{\partial}{\partial \mu_{\dot{\alpha}_1}} \dots \frac{\partial}{\partial \mu_{\dot{\alpha}_h}} f(ix^{\beta\dot{\beta}}\lambda_\beta, \lambda_\beta) \wedge \lambda^\gamma d\lambda_\gamma \tag{4.20}$$

where we have employed homogeneous coordinates λ^α on $\mathbb{CP}^1$. To write the vertex operators on twistor space we must invert this transform, a goal most easily attained with the help of Dirac δ functions. We may define a Dirac δ function on the complex plane $z = x + iy$ via

$$\bar{\delta}(z) = \delta(x)\delta(y) d\bar{z} = \frac{1}{2\pi i} \bar{\partial} \frac{1}{z} \tag{4.21}$$

where the second equality is a consequence of the two-dimensional Green's function for the Laplacian,

$$\delta(x)\delta(y) = \frac{1}{2\pi} \nabla^2 \log(\sqrt{x^2 + y^2}) \tag{4.22}$$

Note that $\bar{\delta}(z)$ has homogeneity -1 in z.

We may similarly define Dirac δ functions on the Riemann sphere of various homogeneities by requiring that the Weyl spinor λ coincides with a fixed spinor λ_a up to scale s_a over which we integrate:

$$\bar{\delta}_h(\lambda, \lambda_a) = \int_{\mathbb{C}} \frac{ds_a}{s_a^{h+1}} \bar{\delta}^{(2)}(\lambda_a - s_a \lambda) \tag{4.23}$$

Consider the mapping $\lambda \to r\lambda$. Changing variables $s_a \to s_a/r$ leaves the integral unchanged up to a factor of r^h, so $\bar{\delta}_h(\lambda, \lambda_a)$ has homogeneity h in λ. Similarly, $\bar{\delta}_h(\lambda, \lambda_a)$ has homogeneity $-h-2$ in λ_a.

It is easy to verify that $\bar{\delta}_h$ behaves as expected; that is, for any function $g(\lambda)$ of homogeneity $-h-2$,

$$g(\lambda_a) = \int_{\mathbb{CP}^1} \bar{\delta}_h(\lambda, \lambda_a) g(\lambda) \wedge \lambda^\gamma d\lambda_\gamma \tag{4.24}$$

We employ an analogous construction to determine the (dual) twistor space wavefunctions corresponding to (4.18), writing

$$V_a(\mu, \lambda) = \int_{\mathbb{C}} \frac{ds_a}{s_a^{h-1}} \; e^{s_a[\mu\lambda_a]} \; \bar{\delta}^{(2)}(\lambda_a - s_a\lambda) \tag{4.25}$$

$$\tilde{V}_a(\tilde{\lambda}, \tilde{\mu}) = \int_{\mathbb{C}} \frac{ds_a}{s_a^{h-1}} \; e^{s_a\langle\tilde{\mu}\lambda_a\rangle} \; \bar{\delta}^{(2)}(\tilde{\lambda}_a - s_a\tilde{\lambda}) \tag{4.26}$$

for a helicity $\pm h/2$ particle respectively, where we have changed notation from f to V to emphasise their role as ambitwistor string vertex operators. For non-abelian theories these operators should be dressed with appropriate color factors. We ignore such complications in this review.

4.3 Scattering Amplitudes

We are now in a position to construct scattering amplitudes with k positive helicity and $n-k$ negative helicity gluons from ambitwistor string theory. The string amplitudes are defined in the usual way, as correlators of vertex operators

$$A_{n,k} = \left\langle \int dz_1 \dots dz_n \; \tilde{V}_1 \dots \tilde{V}_k V_{k+1} \dots V_n \right\rangle \tag{4.27}$$

which may be inserted anywhere on the worldsheet. We may express this as a path integral[52]

$$\begin{aligned} A_{n,k} = \int \frac{\mathcal{D}(\lambda, \tilde{\lambda}, \mu, \tilde{\mu})}{|GL(2;\mathbb{C})|} \int \prod_{i=1}^{k} \frac{dz_i ds_i}{s_i} \bar{\delta}^{(2)}(\tilde{\lambda}_i - s_i \tilde{\lambda}(z_i)) \prod_{j=k+1}^{n} \frac{dz_j ds_j}{s_j} \bar{\delta}^{(2)}(\lambda_j - s_j \lambda(z_j)) \\ \times \exp\left(-[\mu \; \bar{\partial}\tilde{\lambda}] - \langle \tilde{\mu} \; \bar{\partial}\lambda \rangle + \sum_{i=1}^{k} s_i \langle \tilde{\mu} \; \lambda_i \rangle \bar{\delta}(z - z_i) + \sum_{j=k+1}^{n} s_j [\mu \; \tilde{\lambda}_j] \bar{\delta}(z - z_j) \right) \end{aligned} \tag{4.28}$$

in the gauge $a = 0$, where we've used integration by parts on the action. Observe that $(\mu, \tilde{\mu})$ only appear in the exponential, which is exactly linear in these variables. Hence upon integrating out $(\mu, \tilde{\mu})$ we obtain functional δ functions, enforcing

$$\bar{\partial}\lambda = \sum_{i=1}^{k} s_i \lambda_i \bar{\delta}(z - z_i)\,, \quad \bar{\partial}\tilde{\lambda} = \sum_{j=k+1}^{n} s_j \tilde{\lambda}_j \bar{\delta}(z - z_j) \tag{4.29}$$

Now integrating out $(\lambda, \tilde{\lambda})$ amounts to solving these equations, which is trivial in view of the definition (4.21), yielding

$$\lambda(z) = \sum_{i=1}^{k} \frac{s_i \lambda_i}{z - z_i}\,, \quad \tilde{\lambda}(z) = \sum_{j=k+1}^{n} \frac{s_j \tilde{\lambda}_j}{z - z_j} \tag{4.30}$$

With these equalities, the amplitude becomes

$$A_{n,k} = \int \frac{1}{|GL(2;\mathbb{C})|} \prod_{a=1}^{n} \frac{dz_a ds_a}{s_a (z_a - z_{a+1})} \prod_{i=1}^{k} \bar{\delta}^{(2)}(\tilde{\lambda}_i - s_i \tilde{\lambda}(z_i)) \prod_{j=k+1}^{n} \bar{\delta}^{(2)}(\lambda_j - s_j \lambda(z_j)) \tag{4.31}$$

where the additional factors of $(z_a - z_{a+1})$ in the denominator emerge from the color factors we ignored at the end of Section 4.2. To make this expression slightly neater we change homogeneous coordinates on the Riemann sphere, defining

$$\sigma^\alpha = \frac{1}{s}(1, z)\,, \quad (i \; j) = \sigma_i^\alpha \epsilon_{\alpha\beta} \sigma_j^\beta \tag{4.32}$$

whence[53]

$$\lambda(\sigma) = \sum_{i=1}^{k} \frac{\lambda_i}{(\sigma \; \sigma_i)}\,, \quad \tilde{\lambda}(\sigma) = \sum_{j=k+1}^{n} \frac{\tilde{\lambda}_j}{(\sigma \; \sigma_j)} \tag{4.33}$$

and

$$A_{n,k} = \int \frac{1}{|GL(2;\mathbb{C})|} \prod_{a=1}^{n} \frac{d^2\sigma_a}{(a \; a+1)} \prod_{i=1}^{k} \bar{\delta}^{(2)}(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i)) \prod_{j=k+1}^{n} \bar{\delta}^{(2)}(\lambda_j - \lambda(\sigma_j)) \tag{4.34}$$

[52]We use the standard amplitudes notation $\langle a \; b \rangle = \lambda_{a\alpha} \lambda_b^\alpha$ and $[a \; b] = \tilde{\lambda}_a^{\dot{\alpha}} \tilde{\lambda}_{b\dot{\alpha}}$ where raising and lowering of indices is done with the totally antisymmetric tensor.

[53]We have rescaled λ and $\tilde{\lambda}$ by a factor of $\frac{1}{s}$, permitted since they are only projectively meaningful.

This represents the full tree-level S-matrix of Yang-Mills theory in a remarkably compact fashion, as a weighted sum over solutions to the **rational scattering equations**

$$\begin{aligned}\sum_{i=1}^{k} \frac{\lambda_i}{(\sigma_j\ \sigma_i)} &= \lambda_j \quad \text{for } j = k+1, \dots, n \\ \sum_{j=k+1}^{n} \frac{\tilde{\lambda}_j}{(\sigma_i\ \sigma_j)} &= \tilde{\lambda}_i \quad \text{for } i = 1, \dots, k\end{aligned} \tag{4.35}$$

These imply a simpler form by employing partial fractions. We observe

$$\begin{aligned}\left(\sum_{i=1}^{k} \frac{\lambda_i s_i}{z - z_i}\right)\left(\sum_{j=k+1}^{n} \frac{\tilde{\lambda}_j s_j}{z - z_j}\right) &= \sum_{i=1}^{k} \frac{\lambda_i}{z - z_i} \sum_{j=k+1}^{n} \frac{\tilde{\lambda}_j s_i s_j}{z_j - z_i} + \sum_{j=k+1}^{n} \frac{\tilde{\lambda}_j}{z - z_j} \sum_{i=1}^{k} \frac{\lambda_i s_i s_j}{z_i - z_j} \\ &= \sum_{i=1}^{k} \frac{\lambda_i \tilde{\lambda}_i}{z - z_i} + \sum_{j=k+1}^{n} \frac{\lambda_j \tilde{\lambda}_j}{z - z_j}\end{aligned} \tag{4.36}$$

on the support of (4.35). Now defining

$$p^\mu(z) = \prod_{b=1}^{n} (z - z_b) \sum_{c=1}^{n} \frac{p_c^\mu}{z - z_c} \tag{4.37}$$

we see that $p^\mu(z)$ is a simple tensor so $p^2(z) = 0$ identically. In particular, differentation implies the constraint $p(z_a) \cdot p'(z_a) = 0$ for $a = 1, \dots n$. Expanding $p^\mu(z)$ as a degree-$(n-2)$ polynomial (courtesy of momentum conservation) and performing some algebraic gymnastics yields the **scattering equations**

$$f_a(z_1, \dots z_n) = \sum_{b \neq a}^{n} \frac{k_a \cdot k_b}{z_a - z_b} = 0 \tag{4.38}$$

We finish by extracting a specific 3-point amplitude from (4.34), namely that with helicity configuration $(--+)$. This will expose the reader to modern notation for field theory amplitudes, oft encountered in seminars. We make the gauge choice

$$\sigma_1 = (1, 0)\,, \quad \sigma_2 = (0, 1)\,, \quad \sigma_3 = (\tau, \sigma) \tag{4.39}$$

so that

$$(1\ 2) = 1\,, \quad (2\ 3) = -\tau\,, \quad (3\ 1) = \sigma \tag{4.40}$$

The δ functions reduce to

$$\bar{\delta}^{(2)}\left(\tilde{\lambda}_1 + \frac{\tilde{\lambda}_3}{\sigma}\right)\bar{\delta}^{(2)}\left(\tilde{\lambda}_2 + \frac{\tilde{\lambda}_3}{\tau}\right)\bar{\delta}^{(2)}\left(\lambda_3 - \frac{\lambda_1}{\sigma} - \frac{\lambda_2}{\tau}\right) \tag{4.41}$$

The third δ function fixes

$$\sigma = -\frac{\langle 1\ 2\rangle}{\langle 2\ 3\rangle}\,, \quad \tau = \frac{\langle 1\ 2\rangle}{\langle 1\ 3\rangle} \tag{4.42}$$

also contributing a Jacobian factor

$$\begin{vmatrix} \lambda_1/\sigma^2 & 0 \\ 0 & \lambda_2/\tau^2 \end{vmatrix}^{-1} = \frac{\sigma^2 \tau^2}{\langle 1\ 2\rangle} \tag{4.43}$$

The other two δ functions becomes

$$\bar{\delta}^{(2)}\left(\frac{\langle 2\ 1\rangle\tilde{\lambda}_1+\langle 2\ 3\rangle\tilde{\lambda}_3}{\langle 2\ 1\rangle}\right)\bar{\delta}^{(2)}\left(\frac{\langle 1\ 2\rangle\tilde{\lambda}_2+\langle 1\ 3\rangle\tilde{\lambda}_3}{\langle 1\ 2\rangle}\right)=\bar{\delta}^{(2)}\left(\frac{\langle 2|P}{\langle 2\ 1\rangle}\right)\bar{\delta}^{(2)}\left(\frac{\langle 1|P}{\langle 1\ 2\rangle}\right)=\delta^{(4)}(P)\langle 1\ 2\rangle^2 \tag{4.44}$$

where P is the total momentum. Hence (4.34) evaluates to

$$\frac{1}{\sigma\tau}\frac{\sigma^2\tau^2}{\langle 1\ 2\rangle}\delta^{(4)}(P)\langle 1\ 2\rangle^2=\delta^{(4)}(P)\frac{\langle 1\ 2\rangle^3}{\langle 2\ 3\rangle\langle 3\ 1\rangle} \tag{4.45}$$

which may be verified by comparison with any modern textbook.

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